Stability of synchronization in coupled time-delay systems using Krasovskii-Lyapunov theory

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Stability of synchronization in unidirectionally coupled time-delay systems is studied using the Krasovskii-Lyapunov theory. We have shown that the same general stability condition is valid for different cases, even for the general situation (but with a constraint) where all the coefficients of the error equation corresponding to the synchronization manifold are time dependent. These analytical results are also confirmed by the numerical simulation of paradigmatic examples.

Synchronization of coupled chaotic dynamical systems is an active area of research in different branches of science and technology [1,2]. Different kinds of chaos synchronization have been reported both theoretically and experimentally since its discovery in coupled chaotic oscillators [3]. Recent studies on synchronization have been focused on coupled time-delay systems with or without the time-delay coupling because of its intrinsic nature of generating high-dimensional chaotic signals [4] and the ease of experimental realization of these systems [5]. These systems have potential applications in secure communication, cryptography, controlling, long term prediction, optimization of nonlinear system performance, modeling brain activity, pattern recognition, etc. (cf. [4–10]). Specifically, exploiting synchronization, communication with chaotic lasers was demonstrated in [7] and digital information at gigabyte rates was transferred successfully [8] exploiting the time-delayed feedback to generate high-dimensional high-capacity wave forms at high bandwidths. With this kind of applications of high-dimensional chaotic signals of time-delay systems, it becomes extremely important to establish conditions under which synchronized states are asymptotically stable in coupled time-delay systems. For this purpose, the Krasovskii-Lyapunov theory has become an extremely useful tool [9–13].

On the other hand, recently, it has been pointed out that the Krasovskii-Lyapunov theory is not suitable for a more general case where the error equation corresponding to the synchronization manifold is time dependent; especially when all its coefficients are time dependent [11]. In this contribution, we will show that the Krasovskii-Lyapunov theory is not restricted to rather special cases discussed in the literature, so far, but that it can be exploited as a powerful tool in identifying the synchronization thresholds and the stability of synchronization in general coupled time-delay systems. In particular, we will show that the same general stability condition resulting from the Krasovskii-Lyapunov theory is in-

deed valid for the general case, where the coefficients are time dependent. This general situation is very important in many applications such as synchronization via dynamical relaying [14], for instance, the synchronization of neural activity in brain with complex functional architecture has been shown to underlie cognitive acts [15], in dynamically evolving networks [16] such as *ad hoc* networks [17], etc. In particular, we will discuss all the four possible cases that arise due to the nature of the coefficients in the error equation corresponding to the synchronization manifold and show that the same stability condition deduced from the Krasovskii-Lyapunov functional approach is valid for all the cases, subject to certain conditions. We will also confirm these analytical results by numerical analysis using paradigmatic examples.

Consider the following linearly coupled scalar time-delay system,

$$\dot{x}(t) = -ax(t) + bf[x(t-\tau)], \tag{1a}$$

$$\dot{y}(t) = -ay(t) + bf[y(t-\tau)] + K(t)[x(t) - y(t)], \tag{1b}$$

where a and b are positive constants, $\tau > 0$ is the delay time, $K(t)$ is the coupling function between the drive and the response systems, and $f(x)$ is some nonlinear function. Now we can deduce the stability condition for complete synchronization of the general unidirectionally coupled time-delay systems (1). The time evolution of the difference system with the state variable $\Delta = x(t) - y(t)$ [the error equation corresponding to the complete synchronization manifold of the coupled time-delay systems (1)] for small values of Δ can be written as

$$\dot{\Delta} = -[a + K(t)]\Delta + bf'[y(t-\tau)]\Delta_\tau, \quad \Delta_\tau = \Delta(t-\tau). \tag{2}$$

It is to be noted that there arises four cases depending on the nature of the coefficient of the Δ and Δ_τ terms of the above error equation as follows:

(1) Both coefficients of the Δ and Δ_τ terms are time independent.

(2) The coefficient of the Δ term is time independent and that of the Δ_τ term is time dependent.

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(3) The coefficient of the Δ term is time dependent and that of the Δ_τ term is time independent.

(4) Both coefficients of the Δ and Δ_τ terms are time dependent.

The synchronization manifold of the error (2) is locally attracting if the origin of this equation is stable. Following the Krasovskii-Lyapunov theory [9], we define a continuous positive-definite Lyapunov functional of the form

$$V(t) = \frac{1}{2}\Delta^2 + \mu \int_{-\tau}^{0} \Delta^2(t+\theta)d\theta, \quad V(0) = 0 \qquad (3)$$

where μ is an arbitrary positive parameter $\mu > 0$. The derivative of the functional $V(t)$ along the trajectory of the error (2),

$$\frac{dV}{dt} = -[a + K(t)]\Delta^2 + bf'[y(t-\tau)]\Delta\Delta_\tau + \mu\Delta^2 - \mu\Delta_\tau^2, \qquad (4)$$

has to be negative to ensure the stability of the solution $\Delta = 0$. The requirement that $\frac{dV}{dt} < 0$ for all Δ and Δ_τ results in the condition for the stability as

$$a + K(t) > \frac{b^2}{4\mu}f'[y(t-\tau)]^2 + \mu = \Phi(\mu). \qquad (5)$$

Now, $\Phi(\mu)$ as a function of μ for a given $f'(x)$ has an absolute minimum at

$$\mu = \{|bf'[y(t-\tau)]|\}/2, \qquad (6)$$

with $\Phi_{min} = |bf'[y(t-\tau)]|$. Since $\Phi \geq \Phi_{min} = |bf'[y(t-\tau)]|$, from the inequality (5), it turns out that a sufficient condition for the asymptotic stability is

$$a + K(t) > |bf'[y(t-\tau)]|. \qquad (7)$$

It is to be noted that since μ is an arbitrary positive parameter due to the definition of the positive-definite Lyapunov function (3), the above stability condition is valid only when $\mu = \{|bf'[y(t-\tau)]|\}/2$ is a constant, i.e., only when $f'(x)$ is a constant [in other words, when the coefficient of Δ_τ term in the error (2) is time independent, which corresponds to the cases (1) and (3) discussed above]. On the other hand, if $f'(x)$ is time dependent then μ can be obtained alternatively by rewriting Eq. (5) as

$$b^2 f'[y(t-\tau)]^2 < 4\mu[a + K(t) - \mu]$$
$$= -4\{\mu - [a+K(t)]/2\}^2 + [a+K(t)]^2$$
$$\equiv \Psi(\mu). \qquad (8)$$

Now, $\Psi(\mu)$ as a function of μ for a given $f'(x)$ has an absolute maximum at

$$\mu = [a + K(t)]/2, \qquad (9)$$

with $\Psi_{max} = [a+K(t)]^2$. Using this maximum value in the right-hand side of Eq. (8), we obtain the same stability condition as that of Eq. (7) provided $[a+K(t)]/2 > 0$ since $\mu > 0$. Since $a > 0$, this implies $K(t) > -a$, that is, coupling function $K(t)$ should be either positive definite or $|K(t)| > a$ if it is negative. In particular for the case 2, since the coeffi-

cient of the Δ term in the error equation is time independent [which corresponds to the cases (1) and (2) mentioned above], $K(t) = k > -a$ for all t (k: const).

However, there arises an even more general situation where the coefficients of both the Δ and Δ_τ terms are time dependent (case 4), in which case the arbitrary positive parameter μ in the Lyapunov functional has to be chosen as a positive-definite function $\mu = g(t) > 0$ for all t. In this case, one has to consider the derivative of $\mu = g(t)$ also in the derivative of $V(t)$ as follows:

$$\frac{dV}{dt} = -[a + K(t)]\Delta^2 + bf'[y(t-\tau)]\Delta\Delta_\tau + g(t)(\Delta^2 - \Delta_\tau^2)$$
$$+ \dot{g}(t)\int_{-\tau}^{0}\Delta^2(t+\theta)d\theta < 0. \qquad (10)$$

It is known from the Lyapunov functional that the term $\int_{-\tau}^{0}\Delta^2(t+\theta)d\theta$ is positive and let us suppose that $\dot{g}(t) \leq 0$ for all t then for $\dot{V}(t) < 0$ a sufficient condition is that

$$-[a + K(t)]\Delta^2 + bf'[y(t-\tau)]\Delta\Delta_\tau + g(t)(\Delta^2 - \Delta_\tau^2) < 0, \qquad (11)$$

$$-\{[a + K(t)] - b^2 f'[y(t-\tau)]^2/4g(t) - g(t)\}\Delta^2 - g(t)\{\Delta_\tau - bf'[y(t-\tau)]\Delta/2g(t)\}^2 < 0. \qquad (12)$$

The second term in the above equation is negative by assumption of $g(t)$ and hence it follows that

$$b^2 f'[y(t-\tau)]^2 < 4g(t)[a + K(t) - g(t)]$$
$$= -4\{g(t) - [a+K(t)]/2\}^2 + [a+K(t)]^2$$
$$\equiv \Gamma[g(t)]. \qquad (13)$$

Consequently, we obtain the same stability condition as in Eq. (7) with the maximum of Γ, $\Gamma_{max} = [a+K(t)]^2$, occurring at $g(t) = [a+K(t)]/2 > 0$, along with the condition $\dot{g}(t) \leq 0$, that is, $\dot{K}(t) = dK(t)/dt \leq 0$, for all t.

Note that our above analysis holds good in case (4), only for $\dot{K}(t) \leq 0$ and is not valid for $\dot{K}(t) > 0$. For the latter case, we are unable to obtain a sufficiency condition yet. Consequently, the cases (1)–(3) cannot be treated as special cases of the most general case (4) at present.

Thus, we have shown that the same general stability condition [Eq. (7)] is valid for all the four cases that arise in the error (2) corresponding to the synchronization manifold of the unidirectionally coupled time-delay systems with a restriction in case (4).

In this section, we will provide numerical confirmation of the above stability analysis for all the four cases using appropriate nonlinear functional forms $f(x)$ and suitable coupling $K(t)$ in the coupled time-delay systems (1). For this purpose, we will consider the nonlinear functions $f(x)$ as the piecewise linear function, which has been studied in detail recently [12,18,19],
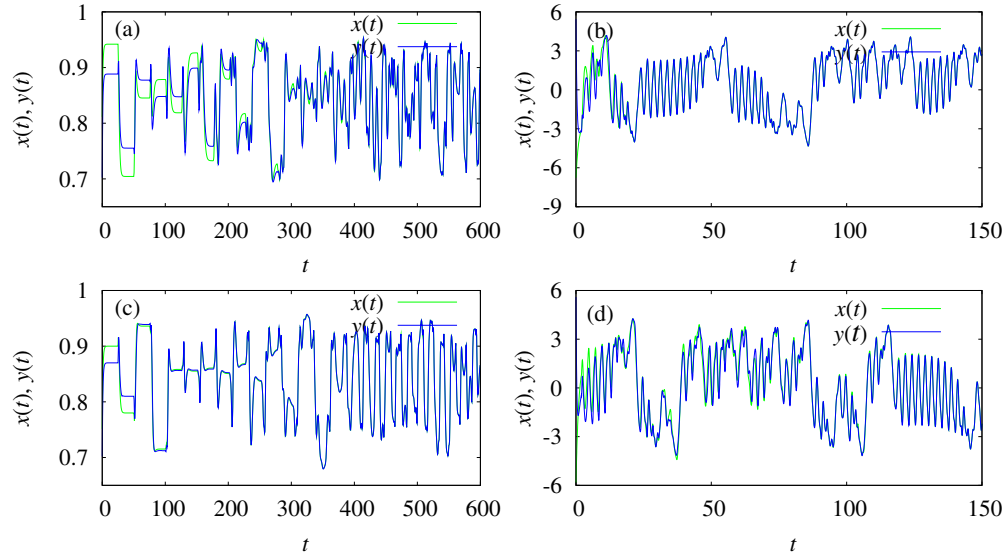
FIG. 1. (Color online) The time trajectory plot of the variables $x(t)$ and $y(t)$ of the coupled time-delay systems (1) indicating complete synchronization between them. (a) Piecewise linear time-delay systems (1) and (14), for the parameters $a=1.0$, $b=1.2$, $\tau=25.0$ and for the constant coupling $k_1=k_2=0.9$. (b) Ikeda time-delay systems (1) and (15), for the parameters $a=1.0$, $b=5$, $\tau=2.0$ along with the constant coupling $k_1=k_2=5.0$. (c) Piecewise linear time-delay system for the same values of the system parameters as in (a) with the square wave coupling rate $k_1=0.9$ and $k_2=1.0$. (d) Ikeda system for the same values of the system parameters and with the square wave coupling rate $k_1=5.0$ and $k_2=6.0$.

$$f(x) = \begin{cases} 0, & x \le -4/3 \\ -1.5x - 2, & -4/3 < x \le -0.8 \\ x, & -0.8 < x \le 0.8 \\ -1.5x + 2, & 0.8 < x \le 4/3 \\ 0, & x > 4/3, \end{cases} \quad (14)$$

and

$$f(x) = \sin[x(t - \tau)], \quad (15)$$

which is the paradigmatic Ikeda model [6]. We have fixed the parameters as $a=1.0$, $b=1.2$ and $\tau=25.0$ for the coupled piecewise linear time-delay system defined by Eqs. (1) and (14), for which the uncoupled systems exhibit a hyperchaotic behavior with nine positive Lyapunov exponents [12,13,18,19]. For the coupled Ikeda systems (1) and (15), the parameters are chosen as $a=1.0$, $b=5.0$ and $\tau=2.0$, where the uncoupled individual Ikeda systems exhibit a hyperchaotic behavior with three positive Lyapunov exponents [13].

We choose the coupling function $K(t)$ as a square wave function represented as [20]

$$K(t) = \{(t_0, k_1), (t_1, k_2), (t_2, k_1), (t_3, k_2), \ldots\}, \quad (16)$$

where $t_j = t_0 + (j-1)\tau_s$, $j \ge 1$ is the switching instant, $k_1 > 0$, $k_2 > 0$ with $k_1 \ne k_2$. For constant coupling $K(t) = k_1 = k_2$. On the other hand, either $k_1 = 0$ or $k_2 = 0$ then the coupling is called an intermittent coupling or control which is now being widely studied in the literature [21].

First, we use the piecewise linear function (14) and the constant coupling $K(t) = k_1 = k_2$. It is clear from the form of the nonlinear function $f(x)$ and the coupling that both coefficients of the Δ and Δ_τ terms in the error (2) are constant

(case 1) and, consequently, μ can either be chosen as $\mu = \{|bf'[y(t-\tau)]|\}/2$ or $\mu = [|a+K(t)|]/2$. The time trajectory of the variables $x(t)$ and $y(t)$ of the coupled piecewise linear time-delay systems (1) and (14) are shown in Fig. 1(a), indicating complete synchronization between them for the coupling strength $k=k_1=k_2=0.9$ satisfying the stability condition $a+k>bf'[y(t-\tau)]=1.5b$. Here, the other system parameters are fixed as noted above.

Next, we analyze the function $f(x)=\sin[x(t-\tau)]$ given by Eq. (15) of the Ikeda system with constant coupling, which corresponds to the case 2 where the coefficient of the Δ_τ term in the error equation is time dependent, while that of the Δ term is still time independent and hence μ can take the form $\mu = [a+K(t)]/2$ with $K(t)>0$. The coupling strength is fixed as $k=k_1=k_2=5.0$ such that the stability condition $a+k>bf'[y(t-\tau)]=b$ is satisfied. The variables $x(t)$ and $y(t)$ of the coupled Ikeda systems (1) and (15) are plotted as a function of time in Fig. 1(b) demonstrating complete synchronization between them.

Again, we consider the piecewise linear function (14) and the same parameter values as in the case 1 but with the square wave coupling $K(t)$ chosen as $k_1=0.9$ and $k_2=1.0$ such that the stability condition (7) is satisfied for all t. The switching instant τ_s between k_1 and k_2 for the square wave coupling rate is fixed as $\tau_s=1.0$. This situation corresponds to the case 3, where the coefficient of the Δ term in the error equation is time dependent, while that of the Δ_τ term is time independent and as a result μ can be fixed as $\mu = \{|bf'[y(t-\tau)]|\}/2$. The time trajectory of the variables $x(t)$ and $y(t)$ are shown in Fig. 1(c), indicating complete synchronization. Note that here that $K(t)>0$ and that the stability condition (7) is indeed satisfied.

Finally, for the more general case where both coefficients of the Δ and Δ_τ term of the error equation are time depen-

dent, $\mu = g(t)$ can be given as $g(t) = [a + K(t)]/2$ for the chosen form of the square wave coupling $K(t)$ with $K(t) > 0$ and $\dot{K}(t) = 0$. Figure 1(d) is plotted for the same values of the system parameters as in Fig. 1(b) with $k_1 = 5.0$, $k_2 = 6.0$ and $\tau_s = 1.0$ satisfying the stability condition (7), indicating complete synchronization between the variables $x(t)$ and $y(t)$.

Asymptotic stability of a synchronized state in a unidirectionally coupled time-delay system is studied using the Krasovskii-Lyapunov theory. We have shown that the same stability condition is valid for all the four cases that arises due to the nature of the coefficients of the Δ and Δ_τ terms in the error equation corresponding to the synchronization manifold. In particular, we have shown that the same general stability condition is valid even for the general case where both coefficients of the Δ and Δ_τ terms in the error equation are time dependent, which is of high importance for various applications. We have also numerically confirmed these results using appropriate examples along with suitable coupling configuration.

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